Matrix product states for quantum metrology

Marcin Jarzyna, Rafał Demkowicz-Dobrzański[1]

[1] *Faculty of Physics, University of Warsaw, ul. Hoża 69, PL-00-681 Warszawa, Poland*

We demonstrate that the optimal states in lossy quantum interferometry may be efficiently simulated using low rank matrix product states. We argue that this should be expected in all realistic quantum metrological protocols with uncorrelated noise and is related to the elusive nature of the Heisenberg precision scaling in presence of decoherence.

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Over the recent years, advancements in quantum engineering have pushed non-classical concepts such as entanglement and squeezing, previously regarded as largely academic topics, close to practical applications. Quantum features of light and atoms helped to improve the performance of measuring devices that operate in the regime were the precision is limited by the fundamental laws of physics [1]. One of the most spectacular examples of practical applications of quantum metrology can be found in gravitational wave detectors [2], were the original idea [3] of employing squeezed states of light to improve the sensitivity of an interferometer has found its full scale realization [4, 5]. No less impressive are the works of the 2012 Nobel Prize winner David Wineland were entangled trapped ions have been demonstrated to provide spectroscopic resolution enhancement crucial for the operation of the atomic clocks [6–8].

When standard sources of laser light are being used, any interferometric experiment may be fully described by treating each photon individually and claiming that *each photon interferes only with itself*. Sensing a phase delay ϕ between the two arms of the interferometer via intensity measurements may be regarded as many independent repetitions of single photon interferometric experiments. N independent experiments results in the data that allows the parameter ϕ to be estimated with error scaling as $1/\sqrt{N}$—the so called standard quantum limit or the shot noise limit. If, however, an experiment cannot be split into N independent processes, as is e.g. the case with the N probing photons being entangled, the above reasoning is invalid and one can in principle achieve the $1/N$ estimation precision—the Heiseberg scaling [9–12]— with the help of e.g. the N00N states [13].

Still, in all realistic experimental setups, decoherence typically makes the relevant quantum features such as squeezing or entanglement die out very quickly. Recently, it has been rigorously shown for optical interferometry with loss [14, 15], as well as for more general decoherence models [16, 17], that if decoherence acts independently on each of the probes one can get at best $c/\sqrt{N}$ asymptotic scaling of error—precision that is better than classical one only by a constant factor c which depends on the type of decoherence and its strength.

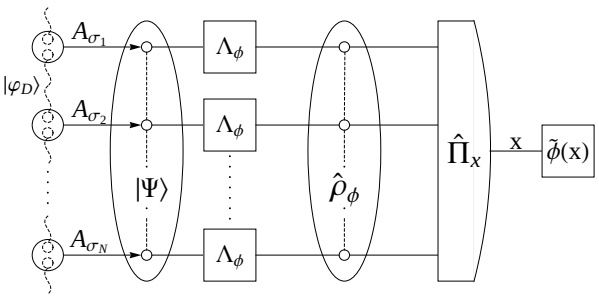

Figure 1. Quantum metrology with matrix product states. N parallel quantum channels act on the input state $|\Psi\rangle$ inscribing parameter value ϕ as well as causing local decoherence on each of the probes. Measurement $\hat{\Pi}_x$ on the output state $\hat{\rho}_\phi$ allows to make an estimate $\tilde{\phi}(x)$ based on the measurement result x. The input state is a matrix product state obtained by the action of maps A_{σ_i} on pairs of virtual D dimensional systems each prepared in a maximally entangled state $|\varphi_D\rangle$.

This implies that in the limit of large number of probes, almost optimal performance can be achieved by dividing the probes into independent groups where only the probes from a given group are entangled among each other. Clearly, the size of the group that is needed to approach the fundamental $c/\sqrt{N}$ bound up to a given accuracy will depend on the strength of decoherence. Nevertheless, irrespectively of how small the decoherence strength is, for N large enough the size of the group will saturate at some point and therefore asymptotically the optimal state may be regarded as only locally correlated.

A natural class of states efficiently representing

locally correlated states are the Matrix Product States (MPS) [18–22], which have proved to be highly successful in simulating low-energy states of complex spin systems. Until now no attempt has been made, however, to employ MPS for quantum metrology purposes. Establishing this connection is the essence of the present paper.

Basic quantum metrology scheme is depicted in Fig. 1. N probe input state $|\Psi\rangle$ travels through N parallel noisy channels Λ_ϕ which action is parameterized by an unknown value ϕ. A measurement $\hat{\Pi}_x$ is performed on the output density matrix $\hat{\rho}_\phi = \Lambda_\phi^{\otimes N}(|\Psi\rangle\langle\Psi|)$ yielding a result x with probability $p(x|\phi) = \mathrm{Tr}(\hat{\rho}_\phi \hat{\Pi}_x)$. The estimation procedure is completed by specifying an estimator function $\tilde{\phi}(x)$. Eventually we are left with the estimated value of the parameter, $\tilde{\phi}$, which in general will be different from ϕ. We denote the average uncertainty of estimation by $\Delta\phi = \sqrt{\langle(\tilde{\phi} - \phi)^2\rangle}$. where the average is performed over different measurement results x. The main goal of theoretical quantum metrology is to find strategies that minimize $\Delta\phi$. For this purpose one has to find the optimal estimator, measurement and input state. This in general is a difficult task.

To simplify the problem one may resort to the quantum Cramer-Rao inequality [23–26]

$$\Delta\phi \geq \frac{1}{\sqrt{kF(\hat{\rho}_\phi)}}, \quad F(\hat{\rho}_\phi) = \mathrm{Tr}(\hat{\rho}_\phi \hat{L}_\phi^2) \qquad (1)$$

that bounds the precision of any unbiased estimation strategy based on k independent repetitions of an experiment. $F(\hat{\rho}_\phi)$ is the Quantum Fisher Information (QFI) written in terms of $\hat{L}_\phi$—the so called symmetric logarithmic derivative (SLD)—defined implicitly as: $2\frac{d\hat{\rho}_\phi}{d\phi} = \hat{L}_\phi\hat{\rho}_\phi + \hat{\rho}_\phi\hat{L}_\phi$. For pure states the formula for QFI simplifies to $F(|\Psi_\phi\rangle) = 4(\langle\dot{\Psi}_\phi|\dot{\Psi}_\phi\rangle - |\langle\dot{\Psi}_\phi|\Psi_\phi\rangle|^2)$, where $|\dot{\Psi}_\phi\rangle = \frac{d|\Psi_\phi\rangle}{d\phi}$. The bound is known to be saturable in the asymptotic limit of $k \to \infty$ in the sense that there exist a measurement and an estimator that yields equality in (1). The main benefit of using QFI is that since it does not depend neither on the measurement nor on the estimator, the only remaining optimization problem is the maximization of $F(\hat{\rho}_\phi)$ over input states.

Since the optimal states in the regime of large number of probes N (not k!) may be regarded as consisting of independent groups, the Cramer-Rao bound may be saturated even for $k = 1$ provided N is large enough [25, 27]. This makes the QFI an even more appealing quantity than in the decoherence-free case were some controversies arise on the practical use of the strategies based on the optimization of the QFI [28, 29].

Maximization of QFI over the most general input states for large N may still be challenging and even if successful might not provide an insight into the structure of the optimal states. This is the place were MPS come in useful.

Here for concreteness we focus on one of the most thoroughly analyzed and relevant model in quantum metrology namely the lossy interferometer. In order to treat uniformly both atom and photon interferometry, we will not specify the nature of the physical systems but will rather refer to abstract two-level probes, with orthogonal sates $|0\rangle$, $|1\rangle$ (e.g. a photon traveling in the lower/upper arm of the interferometer). Consider N distinguishable probes (e.g. photons prepared in different time bins), $|\Psi\rangle = \sum_{i_1 i_2 \ldots i_N=0}^{1} \alpha_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$. The parameter to be estimated is the relative phase delay ϕ a probe experiences being in $|1\rangle$ vs. $|0\rangle$ state. The decoherence mechanism amounts to a loss of probes where each of the probes is lost independently of the others with probability $1 - \eta$. As such, this is an example of a general scheme depicted in Fig. 1. Since the distinguishability of probes offers no advantage for phase estimation [30] it is more convenient to work with indistinguishable probes and move to the occupation number basis where the general N probe state reads $|\Psi\rangle = \sum_{n=0}^{N} \alpha_n |n, N-n\rangle$, and $|n, N-n\rangle$ represents n and $N-n$ probes in states $|0\rangle$ and $|1\rangle$ respectively. Formally speaking, switching from the distinguishable to the indistinguishable probes case is equivalent to restricting the full N probe space to its fully symmetric subspace. The output state $\hat{\rho}_\phi$ can now be written explicitly as:

$$\hat{\rho}_\phi = \sum_{l_0=0}^{N} \sum_{l_1=0}^{N-l_0} p_{l_0 l_1} |\Psi_\phi^{l_0 l_1}\rangle\langle\Psi_\phi^{l_0 l_1}| \qquad (2)$$

where

$$|\Psi_\phi^{l_0 l_1}\rangle = \frac{1}{\sqrt{p_{l_0 l_1}}} \sum_{n=l_0}^{N-l_1} \alpha_n e^{in\phi} \beta_{l_0 l_1}^n |n-l_0, N-n-l_1\rangle$$

$$\beta_{l_0 l_1}^n(\eta) = \sqrt{B_{l_0}^n B_{l_1}^{N-n}}, \; B_l^n = \binom{n}{l} \eta^{N-l}(1-\eta)^l \qquad (3)$$

and $p_{l_0 l_1}$ is a normalization factor which can be interpreted as a probability to lose l_0, l_1 probes in states $|0\rangle$ and $|1\rangle$ respectively.

As the output state $\hat{\rho}_\phi$ is mixed, QFI is not explicitly given in terms of the input state parameters as it requires calculation of the SLD. As was shown in [31], an excellent approximation to QFI is obtained in the form of a weighted sum of QFIs for the pure states entering the mixture (2):

$$F(\hat{\rho}_\phi) \approx \sum_{l_0=0}^{N}\sum_{l_1=0}^{N} p_{l_0,l_1} F(|\Psi_\phi^{l_0,l_1}\rangle). \tag{4}$$

The approximation amounts to a statement that knowing how many probes were lost, additional knowledge of how many of them were lost being in the state $|0\rangle$ and how many being in the state $|1\rangle$ does not improve the estimation precision appreciably. This simplifies the calculations significantly since QFI for pure states has an explicit formula—in our case $F(|\Psi_\phi^{l_0 l_1}\rangle) = 4(\langle\Psi_\phi^{l_0 l_1}|\hat{n}^2|\Psi_\phi^{l_0 l_1}\rangle - |\langle\Psi_\phi^{l_0 l_1}|\hat{n}|\Psi_\phi^{l_0 l_1}\rangle|^2)$ with $\hat{n}$ being the the excitation number operator $\hat{n}|n, N-n\rangle = n|n, N-n\rangle$.

Direct optimization of formula (4) over the input state parameters α_n involves $N+1$ variables. This approach was taken in [30, 31]. Here we want to show that by restricting ourselves to a class of low rank MPS we can achieve almost identical precision with optimization over very few parameters.

Translationaly invariant MPS of N qubits with periodic boundary conditions are defined as

$$|\Psi\rangle_{\mathrm{MPS}} = \frac{1}{\sqrt{\mathcal{N}}}\sum_{i_1\ldots i_N=0}^{1} \mathrm{Tr}(A_{i_1}\ldots A_{i_N})|i_1\ldots i_N\rangle, \tag{5}$$

where two matrices A_i are square complex matrices of dimension $D\times D$ where D is called the bond dimension and $\mathcal{N}$ is the normalization factor, $\mathcal{N} = \sum_{i_1\ldots i_N=0}^{1}\mathrm{Tr}[(A_{i_1}^*\otimes A_{i_1})\ldots(A_{i_N}^*\otimes A_{i_N})]$. In operational terms, a MPS is generated by assuming that each qubit is substituted by a pair of D dimensional virtual systems. Systems corresponding to neighbouring particles are prepared in maximally entangled states $|\varphi_D\rangle = \sum_{j=1}^{D}|j,j\rangle$ (Fig. 1) and maps $A_i = \sum_{\alpha,\beta=1}^{D} A_{i,\alpha,\beta}|i\rangle\langle\alpha,\beta|$ are applied to each pair of the virtual systems yielding eventually the MPS [21]. For the problem considered, the advantage of using MPS stems from the fact that for the bond dimensions D needed to approximate a given state using a MPS will typically be much smaller than N. For the purposes of lossy phase estimation we can restrict ourselves to fully symmetric states for which a sufficient condition is that the matrices A_0, A_1 be diagonal. Even though for a given D a larger class of symmetric states could be generated with more general matrices, we have checked that this does not provide any computational benefit in our case. As a results the MPS are parameterized with $2D$ complex numbers instead of $N+1$ needed to specify a general symmetric state of N qubits. In the standard occupation number basis the MPS has the form:

$$|\Psi\rangle_{\mathrm{MPS}} = \frac{1}{\sqrt{\mathcal{N}}}\sum_{n=0}^{N}\sqrt{\binom{N}{n}}\mathrm{Tr}(A_0^n A_1^{N-n})|n, N-n\rangle. \tag{6}$$

Whenever possible one should compute the quantities of interest directly on the A_i matrices. Fortunately, thanks to the simple form of Eq. (4) it is indeed possible and there is no need to go back to the standard state description as would be the case with the exact formula (1).

Fig. 2a illustrates the precision obtained using MPS for the case of relatively small losses $\eta = 0.9$. As one can see, the MPS approximation is excellent. In particular, the upper-right inset shows that already $D = 5$ is sufficient to obtain less than 1% discrepancy for $N \leq 500$. We have confirmed this observation for different η and observed that for higher losses (lower η) lower D are required to obtain a given level of approximation for a particular N—an effect that should be much more spectacular for larger N reflecting the fact that stronger decoherence diminishes the role of quantum correlations.

Moreover we have observed that optimal matrices A_0, A_1 have the same diagonal values which are ordered complementarily—the highest in A_0 is paired with lowest one in A_1 etc. The higher is N the closer the diagonal values approach each other as can be seen from the lower-left inset on Fig. (2). This confirms the intuition that with increasing N, the optimal states are becoming less distinct from the product state—all diagonal values of A_i equal.

The peculiarity of phase estimation is that in the decoherence-free case optimal QFI is achieved for the N00N state which, even though has non-local correlations, is an example of an MPS with $D = 2$. This makes the MPS capable of approximating the optimal states very well even for low loss and small N [$N \lesssim 1/(1-\eta)$]—an ability that in general will not hold for other estimation problems.

Until now we have focused our analysis only on the QFI and avoided discussion of any particular measurement scheme. Taking less fundamental but more operational approach, one may consider a concrete measurement scheme were a particular observable $\hat{O}$ is being measured. Simple error-propagation

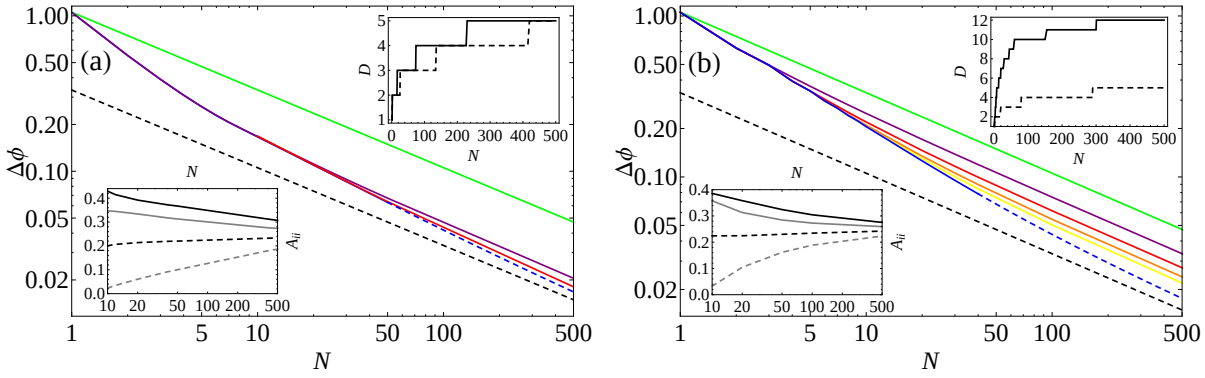

Figure 2. (color online) Log-Log plots of the phase estimation precision with losses ($\eta = 0.9$) as a function of number of probes N optimized over input states in (a) the QFI approach were $\Delta\phi = \frac{1}{\sqrt{F}}$ with F given by Eq. (4) and (b) the Ramsey interferometry with $\Delta\phi$ given by Eq. (7). The color curves correspond to matrix product states of different bond dimensions D: $D = 1$ (green, no correlations: $\Delta\phi = 1/\sqrt{\eta N}$), $D = 2$ (purple), $D = 3$ (red), $D = 4$ (orange), $D = 5$ (yellow). The blue curves correspond to the optimal states obtained with brute force optimization, with extrapolation for higher N (dashed) obtained using scaling formulas of [14] (a) and optimization over the spin-squeezed states [33] (b). The black line represents the asymptotic bound $\Delta\phi = \sqrt{\frac{1-\eta}{\eta N}}$ [14–17]. The upper-right insets depict the bond dimension D needed to reach the optimal precision with at most 1% discrepancy: $\eta = 0.9$ (solid), $\eta = 0.3$ (dashed). Lower-left insets present normalized absolute values of diagonal matrix A_i elements for bond dimension $D = 4$ and $\eta = 0.9$.

formula for ϕ yields $\Delta\phi = \Delta\hat{O}/|\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}\phi}|$. In the Ramsey spectroscopy setup [32], or equivalently in the Mach-Zehnder interferometer with photon number difference measurement, one effectively measures a component of the total angular momentum operator $\hat{J}$ of N spins 1/2—if a qubit $|0\rangle$, $|1\rangle$ is treated as a spin 1/2 particle [11]. If the phase dependent rotation $\hat{U} = e^{i\phi\hat{J}_z}$ is being sensed by the measurement of the $\hat{J}_x$ observable, the explicit formula for estimation uncertainty at the optimal operation point $\phi = 0$ calculated for $\hat{\rho}_\phi$ from Eq. (2) reads

$$\Delta\phi = \sqrt{\frac{\Delta^2\hat{J}_x}{\langle\hat{J}_y\rangle^2} + \frac{1-\eta}{\eta}\frac{N}{4\langle\hat{J}_y\rangle^2}}. \qquad (7)$$

Search for the optimal state amounts to minimizing the above quantity. Since it depends only on first and second moments of $\hat{J}$ it is simple to implement numerically using MPS. Results are presented in Fig. 2b.

It is clear that MPS are capable to capture the essential feature of the optimal states—the squeezing of the $\hat{J}_x$—with relatively low bond dimensions D. Moreover, the upper-right inset indicates that the required bond dimension D is reduced much more significantly with increasing decoherence strength than in the QFI approach. The lower-left inset confirms again that the structure of the optimal states gets closer to the product state structure with increasing N.

In summary, we have shown that MPS are very well suited for achieving the optimal performance in realistic quantum metrological setups and may reduce the numerical effort while searching for the optimal estimation strategies. Even though we have based our presentation on a single model of lossy phase estimation we predict this conclusions to be valid in all metrological setups were decoherence makes the asymptotic Heisenberg scaling unachievable—the intuitive argument being that no large scale strong correlations are needed to reach the optimal performance. An intriguing open question remains: is it possible, as it is in many-body physics problems, to obtain an exponential reduction in numerical complexity thanks to the use of MPS. This is not possible when the optimal states are known to be symmetric, as in the lossy phase estimation, since then the Hilbert space dimension grows only linearly with N. In problems, however, were distinguishability of probes is essential as e.g. in reference frame estimation [34], MPS might demonstrate their full potential.

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